Drag on an axially symmetric body vibrating slowly along its axis in a viscous fluid

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Let D_0 be the Stokes drag on an axially symmetric body moving parallel to its axis with velocity U_0 through an unbounded fluid. The drag D experienced by the same body oscillating with velocity $U = U_0 e^{i\sigma t}$ along its axis in the unbounded fluid is given by the expression

$$\frac{D}{D_0} = \left\{1 + \frac{D_0}{6\sqrt{(2)}\,\pi\mu a U_0}(1+i)\,M + O(M^2)\right\} e^{i\sigma t},$$

where a is any characteristic particle dimension and

$$M^2 = a^2\sigma\rho/\mu$$

is a dimensionless number. The part of this drag formula which gives the energy dissipation is calculated for bodies of various shapes.

1. Introduction

The problem of slow vibrations of a class of axially symmetric bodies in an incompressible viscous fluid has been discussed by the author (1955). With the exception of a sphere and a cylinder, the results turn out to be in the form of certain series of complicated wave functions which for the lack of necessary tables cannot be calculated numerically. Recently certain expansion techniques have been developed by Lagerstrom & Cole (1955) and Proudman & Pearson (1957) for discussing Stokes and Oseen flows. Their arguments have been used and clarified by various other workers in the field. For example, Chang (1960) has studied Stokes flow of a conducting fluid past an axially symmetric body. We make use of this method for solving our problem for small Reynolds numbers. The following discussion is restricted to a less formal summary of those aspects of the techniques which are directly relevant to the present problem. For a detailed account the reader is referred to the above papers.

Brenner (1961) and Brenner & Cox (1963) have also used these expansions to find the general formula for the Oseen drag. In their analysis they have shown that the drag formula found for an axially symmetric body remains valid even for a class of bodies which are not axially symmetric. By arguments similar to those given by them and from the analysis of Proudman & Pearson (1957, p. 245) it can be shown that the formula presented in this paper remains true for a body of an arbitrary shape.

2. Equations of motion

We consider the unsteady flow of an incompressible, viscous fluid when an axially symmetric body of finite size is oscillating longitudinally along its axis of symmetry. It oscillates with velocity $U = U_0 e^{i\sigma t}$ about its centre of inertia as the mean position. Such a motion is governed by Navier–Stokes equations:

$$\nabla . \mathbf{v} = 0, \tag{1}$$

$$-\nabla p + \mu \nabla^2 \mathbf{v} - \rho(\mathbf{v} . \nabla \mathbf{v} + \partial \mathbf{v}/\partial t) = 0, \tag{2}$$

where $\mathbf{v}$ is the velocity vector, p is the pressure, ρ is the density and μ is the viscosity. Both ρ and μ are constants.

Let a be any characteristic length. The space co-ordinates may then be made non-dimensional with the factor a^{-1}. The time t may be made non-dimensional with the help of the circular frequency σ. Thus

$$\mathbf{r}' = \mathbf{r}/a, \quad t' = \sigma t,$$

where $\mathbf{r} = (x, y, z)$. The pressure and the velocity may be made dimensionless by the transformations

$$p' = (a/\mu U_0)\, p, \quad \mathbf{v}' = \mathbf{v}/U_0. \tag{3}$$

The equations (1) and (3) then reduce (*after omitting the primes*) to

$$\nabla . \mathbf{v} = 0, \tag{4}$$

and

$$-\nabla p + \nabla^2 \mathbf{v} - R\mathbf{v} . \nabla \mathbf{v} - M^2(\partial \mathbf{v}/\partial t) = 0, \tag{5}$$

where $R = U_0 a\rho/\mu$ is the Reynolds number and $M^2 = a^2\sigma\rho/\mu$ is another dimensionless number. We assume that $a\sigma$ and U_0 are of the same order of magnitude. Thus, by suitable choice of a, we can make $M^2 = R$ and (5) then becomes

$$-\nabla p + \nabla^2 \mathbf{v} - M^2(\mathbf{v} . \nabla \mathbf{v} + (\partial \mathbf{v}/\partial t)) = 0. \tag{6}$$

Note that the boundary conditions (non-dimensional) are

$$\left.\begin{aligned} \mathbf{v} &= 0, \quad \text{at infinity,} \\ \text{and} \qquad \mathbf{v} &= \mathbf{I}\, e^{it}, \quad \text{at the body,} \end{aligned}\right\} \tag{7}$$

where $\mathbf{I}$ is the unit vector along the direction of the oscillations, i.e. the axis of symmetry of the body, taken to be the x-axis in this analysis. Physically the problem is the same if we set $\mathbf{v} = 0$ at the boundary and $\mathbf{v} = \mathbf{I}\, e^{it}$ at infinity. *We shall assume the latter.*

3. The inner and outer expansions

The inner expansions of the exact solutions of the equations (4) and (6) are of the form

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{h}^{(0)}(x, y, z, t) + M\mathbf{h}^{(1)}(x, y, z, t) + M^2\mathbf{h}^{(2)}(x, y, z, t) + \ldots, \\ p &= p^{(0)}(x, y, z, t) + Mp^{(1)}(x, y, z, t) + M^2 p^{(2)}(x, y, z, t) + \ldots, \end{aligned}\right\} \tag{8}$$

where both the series have complex components. When we insert these expansions into equations (4) and (6) we obtain

$$O(1):\quad \nabla.\mathbf{h}^{(0)} = 0, \qquad -\nabla p^{(0)} + \nabla^2\mathbf{h}^{(0)} = 0; \tag{9}$$

$$O(M):\quad \nabla.\mathbf{h}^{(1)} = 0, \qquad -\nabla p^{(1)} + \nabla^2\mathbf{h}^{(1)} = 0; \tag{10}$$

$$O(M^2):\quad \nabla.\mathbf{h}^{(2)} = 0, \qquad -\nabla p^{(2)} + \nabla^2\mathbf{h}^{(2)} - (\mathbf{h}^{(0)}.\nabla\mathbf{h}^{(0)} + \partial\mathbf{h}^{(0)}/\partial t) = 0; \tag{11}$$

and so on. The equations (9) and (10) are identical with the steady Stokes-flow equations.

For the outer expansions we set

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{I}\,e^{it} + M\mathbf{g}^{(1)}(\tilde{x},\tilde{y},\tilde{z},t) + M^2\mathbf{g}^{(2)}(\tilde{x},\tilde{y},\tilde{z},t) + \ldots, \\ p &= -Mi\tilde{x}\,e^{it} + M^2\tilde{p}^{(1)}(\tilde{x},\tilde{y},\tilde{z},t) + M^3\tilde{p}^{(2)}(\tilde{x},\tilde{y},\tilde{z},t) + \ldots, \end{aligned}\right\} \tag{12}$$

where the first term in the first of equations (12) corresponds to the free-stream velocity $\mathbf{I}\,e^{it}$ at infinity and the independent variables are now

$$\tilde{x} = Mx, \quad \tilde{y} = My, \quad \tilde{z} = Mz. \tag{13}$$

By insertion of these equations in the equations (4) and (6) we obtain the equations for $\tilde{p}_1$ and $\mathbf{g}^{(1)}$ as

$$\tilde{\nabla}.\mathbf{g}^{(1)} = 0, \quad -\tilde{\nabla}\tilde{p}^{(1)} + \tilde{\nabla}^2\mathbf{g}^{(1)} - \partial\mathbf{g}^{(1)}/\partial t = 0. \tag{14}$$

In this article we plan to obtain the solution to our problem correct to order M. As such it suffices to consider the equations (9), (10) and (14). Now assume that $\mathbf{v} = \mathbf{v}_0 e^{it}$, $p = p_0 e^{it}$, and that the same relationship holds for the quantities $\mathbf{h}^{(0)}$, $\mathbf{h}^{(1)}$, $p^{(0)}$, $p^{(1)}$, $\mathbf{g}^{(1)}$ and $\tilde{p}^{(1)}$. When we make these substitutions and *drop the zero subscript*, the equations (9), (10) and the first of (14) are unchanged, while the second of equations (14) becomes

$$-\tilde{\nabla}\tilde{p}^{(1)} + \tilde{\nabla}^2\mathbf{g}^{(1)} - i\mathbf{g}^{(1)} = 0. \tag{15}$$

The boundary conditions become

$$\mathbf{v} = \mathbf{I} \text{ as } r \to \infty, \quad \mathbf{v} = 0 \text{ at the body.} \tag{16}$$

There remains the problem of determining the proper boundary conditions for the individual terms of the inner and outer expansions. Since the inner expansions are not valid at infinity, the first boundary condition in (16) does not in general hold for the inner expansions. For a similar reason, the second boundary condition in (16) does not hold for the outer expansions. These boundary conditions are replaced by the matching conditions, which have the requirements that the inner and outer expansions should agree term by term in their common domain of validity. This shall happen for some intermediate orders of r, namely $r = O(M^{-\alpha})$ where $0 < \alpha < 1$.

4. The first-order inner and outer solutions

The first term in the outer expansion (12) for $\mathbf{v}$ is the free-stream velocity $\mathbf{v} = \mathbf{I}$. This term may be obtained by the following reasoning. Assume the solid is a sphere of radius a. Then the boundary of the solid is given in outer variables by $\tilde{r} = M$ and, in the limit $M \to 0$, the body shrinks to a point. A point cannot cause a finite disturbance in the fluid, hence the value of $\mathbf{v}$ will tend to the free-stream velocity $\mathbf{I}$.

For the inner solutions, the no-slip boundary conditions are valid. Thus

$$\mathbf{h}^{(i)} = 0 \quad \text{at the body.} \tag{17}$$

By the matching conditions, $\mathbf{h}^{(0)}$ must agree for large values of r with the leading terms of the outer expansion, and hence

$$\mathbf{h}^{(0)} \to \mathbf{I} \quad \text{as} \quad r \to \infty. \tag{18}$$

The equations (9), (17) and (18) show that the solutions for $\mathbf{h}^{(0)}$ and $p^{(0)}$ are simply the solutions of the steady Stokes flow problem. For a sphere, such solutions are

$$\left.\begin{aligned} \mathbf{h}^{(0)} &= \mathbf{I} - \frac{3}{2}\left(\frac{\mathbf{I}}{r} - \nabla\frac{x}{2r}\right) + \frac{1}{4}\nabla\frac{\partial}{\partial x}\frac{1}{r}, \\ p^0 &= -\frac{3x}{2r^3}. \end{aligned}\right\} \tag{19}$$

The drag on the sphere of radius a in physical units is $6\pi\mu U_0 a$.

For large values of r, the asymptotic expansions of the steady Stokes flow solutions for axially symmetric bodies of any shape are given as (Payne & Pell 1960)

$$\left.\begin{aligned} \mathbf{h}^{(0)} &= \mathbf{I} - \frac{D_0}{4\pi}\left(\frac{\mathbf{I}}{r} - \nabla\frac{x}{2r}\right) + O\left(\frac{1}{r^2}\right), \\ p^0 &= -\frac{D_0 x}{4\pi r^3} + O\left(\frac{1}{r^3}\right), \end{aligned}\right\} \tag{20}$$

where D_0 is the non-dimensional drag on the body in steady Stokes flow. Now if one defines $r_\alpha = M^\alpha r$, $0 < \alpha < 1$, then the relations (20) become

$$\left.\begin{aligned} \mathbf{h}^{(0)}(x_\alpha, r_\alpha; M) &= \mathbf{I} - \frac{D_0}{4\pi}\left(\frac{\mathbf{I}}{r_\alpha} - \nabla_\alpha\frac{x_\alpha}{2r_\alpha}\right)M^\alpha + O(M^{2\alpha}), \\ p^0(x_\alpha, r_\alpha; M) &= -\frac{D_0 x_\alpha}{4\pi r_\alpha^3}M^{2\alpha} + O(M^{3\alpha}). \end{aligned}\right\} \tag{21}$$

Let us now find the first-order outer solutions. The outer solutions $\mathbf{g}^{(1)}$ and $\tilde{p}^{(1)}$ are determined from the equations (14) and (15) with the boundary conditions $\mathbf{g}^{(1)} \to 0$ as $\tilde{r} \to \infty$ and subject to the matching condition as stated above. To find the required solutions set

$$\mathbf{g}^{(1)} = D_0(G_1, G_2, G_3), \quad \tilde{p}^{(1)} = D_0 P. \tag{22}$$

The contribution for the drag comes only from G_1 which is given by the expression:

$$G_1 = \frac{1}{4\pi}\left[\left(3\frac{\tilde{x}^2}{\tilde{r}^5} - \frac{1}{\tilde{r}^3}\right)\{1 - e^{-\gamma\tilde{r}}(1+\gamma\tilde{r})\}\gamma^2 + e^{-\gamma\tilde{r}}\left(\frac{\tilde{x}^3}{\tilde{r}^3} - \frac{1}{\tilde{r}}\right)\right], \tag{23}$$

where $\gamma^2 = i$, or $\gamma = (1+i)/\sqrt{2}$. The corresponding value of P is $-\tilde{x}/4\pi\tilde{r}^3$.

When we rewrite (23) in the variables r_α and expand in powers of M we get

$$MG_1 = -\frac{1}{4\pi}\left\{\frac{1}{r_\alpha} - \frac{\partial}{\partial x_\alpha}\left(\frac{x_\alpha}{2r_\alpha}\right)\right\}M^\alpha + \frac{1+i}{6\sqrt{(2)}\pi}M + O(M^{2-\alpha}). \tag{24}$$

The leading term in the outer expansion which is not matched by the inner expansion is now of order $O(M)$. From (24) it follows that the second-order inner solution $\mathbf{h}^{(1)}$, which satisfies the Stokes equations (10), should satisfy the boundary condition $\mathbf{h}^{(1)} = 0$, at the body and the condition $\mathbf{h}^{(1)} = [D_0(1+i)/6\sqrt{(2)}\pi]\mathbf{I}$ as $r \to \infty$. Such a solution is easily seen to be

$$\mathbf{h}^{(1)} = \frac{D_0}{6\sqrt{(2)}\,\pi}(1+i)\,M\mathbf{h}^{(0)}. \tag{25}$$

To order M, the inner solution of the velocity field is then

$$\mathbf{v} = \left\{1+\frac{D_0}{6\sqrt{(2)}\,\pi}(1+i)\,M\right\}\mathbf{h}^{(0)}+O(M^2). \tag{26}$$

Body	D_0	D_f
Hemispherical cup	$(3\pi+8)\mu a U_0$	$(3\pi+8)\mu a U\left(1+\frac{3\pi+8}{6\sqrt{(2)}\,\pi}M\right)$
Circular disk (broadside to the stream)	$16\mu a U_0$	$16\mu a U\left(1+\frac{4\sqrt{2}}{3\pi}M\right)$
Circular disk (edge-on to the stream)	$\frac{32}{3}\mu a U_0$	$\frac{32}{3}\mu a U\left(1+\frac{8\sqrt{3}}{9\pi}M\right)$
Sphere	$6\pi\mu a U_0$	$6\pi\mu a U\left(1+\frac{1}{\sqrt{2}}M\right)$
Prolate spheriod	$8\pi\delta\mu a U_0$	$8\pi\delta\mu a U\left(1+\frac{2\sqrt{2}}{3}\delta M\right)$
Oblate spheroid	$8\pi\beta\mu a U_0$	$8\pi\beta\mu a U\left(1+\frac{2\sqrt{2}}{3}\beta M\right)$

Here δ and β are constants related to the geometry of the spheroid (Payne & Pell 1960). The value of D_0 for a hemispherical cup is taken from a recent paper (Collins 1963).

TABLE 1

Therefore the drag on the body is given by the formula (*reverting to physical units and restoring the term* $e^{i\sigma t}$)

$$D = D_0\left\{1+\frac{D_0}{6\sqrt{(2)}\,\pi\mu a U_0}(1+i)\,M+O(M^2)\right\}e^{i\sigma t}. \tag{27}$$

The part

$$D_f = D_0\left\{1+\frac{D_0}{6\sqrt{(2)}\,\pi\mu a U_0}M+O(M^2)\right\}e^{i\sigma t} \tag{28}$$

gives the frictional force while the other part

$$D_m = iD_0\left\{\frac{D_0}{6\sqrt{(2)}\,\pi\mu a U_0}M+O(M^2)\right\}e^{i\sigma t} \tag{29}$$

is $\frac{1}{2}\pi$ out of phase with the velocity of the body and describes the virtual mass of the surrounding fluid associated with the motion.

For a sphere, the formula (27) becomes (to order M)

$$D = 6\pi\mu a U\left\{1+\frac{M}{\sqrt{2}}(1+i)\right\}. \tag{30}$$

This agrees with the known result (Lamb 1945, p. 644, equation (26), or Landau & Lifshitz 1959, p. 96, equation (3)). *In fact the frictional force D_f agrees completely because the known solution for D_f does not contain terms of higher than order M.* Note that U in the formula (30) is $U_0 e^{i\sigma t}$.

The energy dissipation arises only from the part D_f of the drag. The part D_m does not involve any dissipation of energy.

Table 1 compares the values of D_f with D_0 for bodies of various shapes.

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